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Phase transition in a q-deformed Lipkin model

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Abstract. Assuming q-deformed commutation relations for the fermions, an extension of the standard Lipkin Hamiltonian is presented. The usual quasi-spin representation of the standard Lipkin model is also obtained in this q-deformed framework. A variationally obtained energy functional is used to analyse the phase transition associated with the spherical symmetry breaking. The only phase transitions in this q-deformed model are of second order. As an outcome of this analysis a critical parameter is obtained which is dependent on the deformation of the algebra and on the number of particles.

1. Introduction

In the last decade a great effort has been devoted to the development and understanding of deformed algebras, although in many cases their direct physical interpretation is incomplete or even completely lacking. In some cases, like the XXZ-model where the ferromagnetic/antiferromagnetic nature of a spin-$\frac{1}{2}$ chain of length N can be simulated through the introduction of a q-deformed algebra, or the rotational bands in deformed nuclei which can be fitted instead of using a variable moment of inertia (VMI-model) via a q-rotor Hamiltonian, the physical meaning of such a deformation is clearly established. From the original studies, which appeared in connection with problems related to solvable statistical mechanics models [1] and quantum inverse scattering theory [2], a solid development has emerged which encompasses nowadays various branches of mathematical problems related to physical applications, such as deformed superalgebras [3], knot theories [4], non-commutative geometries [5] and so on. In this context, the introduction of a q-deformed bosonic harmonic oscillator, derived in such a way to pass from a $su(2)$ symmetry, originally present in the non-deformed case, to a $su_q(2)$ one, gave origin to new commutation relations which have been extensively studied in several papers [6, 7], all these results being unambiguously obtained due to the underlying $sl_q(2)$ structure [8].

The many-body problem is another mainstream area in physics and, in all its complexity, it calls for the use of approximate methods or the development of simple solvable models which should entail most of the relevant physics combined with a technically simple treatment [9]. A long heritage of such models is available in the nuclear physics literature, among which the Lipkin model [10] has been extensively used as a laboratory to test approximate methods and to point out the main features of the many-body systems.

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Nowadays, an important problem is to understand how the basic characteristics and the general behaviour of many-body systems are modified when the underlying fermionic algebra is deformed. The use of q-deformed algebra in the description of some many-body systems has lead to the appearance of new features when compared to the non-deformed case. In this connection we mention some examples: (i) in the q-oscillator many-body problem [7] it was shown that, when promoting the symmetries of the standard oscillator system to q-symmetries, the spectrum of the system is found to exhibit interactions between the levels of the individual oscillators; (ii) the revivals phenomenon present in the Jaynes–Cummings model [11] disappears when the original $su(2)$ symmetry is deformed; (iii) an extensive study of a deformed collective Lipkin Hamiltonian was performed and the q-deformed second-order phase transition was found to be suppressed [12]. The second-order phase transition associated with the spherical symmetry breaking in the quasi-spin space [13] for this deformed model was also discussed in the q-coherent states framework [14].

In the present paper we return to the original fermionic Lipkin Hamiltonian and extend it to a new one written in terms of q-deformed fermionic operators. Our main goals in such a study are: (i) to try to get some idea on the influence of this q-deformation and (ii) to investigate if new physical phenomena comes out via the introduction of the deformed algebra. Similarly to the extraction of a collective Lipkin Hamiltonian in terms of $su(2)$ quasi-spin operators from the original fermionic one, we show here that such a construction in terms of $su_q(2)$ quasi-spin operators written now in terms of q-deformed fermionic operators, is still valid. This new Hamiltonian is very different from that discussed in [12, 14] in that the mean-field term embodies now the effects of the deformation of the algebra giving rise to a change in the one-body energy spectrum in a similar way to the q-oscillator many-body problem [7].

We study the only phase transitions in this q-deformed model, which are of second order, following Holzwarth [13], i.e. the spherical symmetry breaking in the quasi-spin space, with this new collective deformed Hamiltonian. q-coherent states are used to define θ and φ as collective variables in terms of which the phase transition is analysed through the behaviour of the variationally obtained ground-state energy.

This paper is organized as follows. In section 2 we lay the basis of the q-fermionic extension of the standard Lipkin model (SLM) and the new collective deformed Lipkin Hamiltonian is constructed. Section 3 contains the basic definitions of the q-coherent states and the derivation of the ground-state energy functional. Finally, in section 4 the main results and conclusions are presented.

2. The q-deformed Lipkin model

Since the standard Lipkin model has been widely studied in the literature [15], we will only present here its main features.

In the SLM a system of N fermions is distributed in two N-fold degenerated levels. These two levels are distinguished by a quantum number σ whose values $+1$ and -1 refer to the upper and lower levels respectively; they are separated by an energy gap ϵ. The degeneracy of each level is taken care of by a quantum number p with values ranging from 1 to N.

The SLM Hamiltonian is written as

$$H = \frac{\epsilon}{2}\sum_{p,\sigma}\sigma a^{\dagger}_{p,\sigma}a_{p,\sigma} + \frac{V}{2}\sum_{p,p',\sigma} a^{\dagger}_{p,\sigma}a^{\dagger}_{p',\sigma}a_{p',-\sigma}a_{p,-\sigma}. \tag{2.1}$$

The main advantage of this model, as was originally shown by Lipkin [10], is that it is exactly soluble, its collective excitations being more clearly studied in the $su(2)$ quasi-spin formalism. As a result, a second-order phase transition is obtained from the calculated collective spectrum in the $N \to \infty$ limit [16].

Recently [12], the SLM has been studied in the q-deformed $su(2)$ quasi-spin formalism where q is the deformation parameter of the algebra. The usual $su(2)$ algebra is recovered when $q \to 1$. In this paper the phase transition was analysed in the q-deformed context and the main result of the analysis was the suppression of the phase transition as q increases. A problem akin to this was studied in another paper by one of the authors using deformed $su(2)$ coherent states in a variational approach [14]. On the other hand, a similar result was obtained in a study of the revivals which appear in the Jaynes–Cummings model [11], which are suppressed in the deformed case [17].

From a more fundamental point of view, the question on the possibility of constructing a q-deformed Lipkin Hamiltonian from the basic fermion operators is a very important one. The hint to answer this question is the study of a system of M bosonic harmonic oscillators developed by Floratos [7], where the full Hamiltonian is not just the sum of individual oscillators but rather a sum of terms involving coproducts, the construction being necessary in order to preserve the $U(M)$ symmetry when the algebra is deformed. Although we have not mathematically proved that the derived expressions satisfy a genuine fermionic coproduct, the adopted form is important to obtain the q-deformed quasi-spin operators, as will be seen later. Following the idea of Floratos, the deformed fermionic Lipkin Hamiltonian, with the deformation parameter expressed as $q = \exp\gamma$, is given by

$$H = \mathcal{H}_0 + \mathcal{H}_1 \tag{2.2}$$

with

$$\mathcal{H}_0 = \frac{\epsilon}{2}\sum_p \mathrm{e}^{2\gamma\mathcal{A}_<(p)}\{\mathcal{H}_{p\uparrow}\cosh\gamma h_{p\downarrow} - \cosh\gamma h_{p\uparrow}\mathcal{H}_{p\downarrow}\}\mathrm{e}^{-2\gamma\mathcal{A}_>(p)} \tag{2.3}$$

$$\mathcal{H}_1 = \frac{V}{2}\left[\left(\sum_p \mathrm{e}^{\gamma\mathcal{A}_<(p)}a^\dagger_{p\uparrow}a_{p\downarrow}\mathrm{e}^{-\gamma\mathcal{A}_>(p)}\right)^2 + \left(\sum_p \mathrm{e}^{\gamma\mathcal{A}_<(p)}a^\dagger_{p\downarrow}a_{p\uparrow}\mathrm{e}^{-\gamma\mathcal{A}_>(p)}\right)^2\right] \tag{2.4}$$

where

$$\mathcal{H}_{p\uparrow(\downarrow)} = \tfrac{1}{2}\left(a^\dagger_{p\uparrow(\downarrow)}a_{p\uparrow(\downarrow)} - a_{p\uparrow(\downarrow)}a^\dagger_{p\uparrow(\downarrow)}\right) \tag{2.5}$$

$$\mathcal{A}_<(p) = \tfrac{1}{2}\sum_{p'<p}(h_{p'\uparrow} - h_{p'\downarrow}) \tag{2.6}$$

$$\mathcal{A}_>(p) = \tfrac{1}{2}\sum_{p'>p}(h_{p'\uparrow} - h_{p'\downarrow}) \tag{2.7}$$

and

$$h_{p\uparrow(\downarrow)} = a^\dagger_{p\uparrow(\downarrow)}a_{p\uparrow(\downarrow)} - \tfrac{1}{2}. \tag{2.8}$$

In the above expressions the fermionic operators obey q-deformed anticommutation relations below [3, 18]; however, it is worth noting that the adopted q-deformed fermionic extension of the usual anticommutation relations are by no means unique in the literature [19].

$$a_{p\sigma}a^\dagger_{p\sigma} + q a^\dagger_{p\sigma}a_{p\sigma} = q^{\hat{n}_{p\sigma}} \tag{2.9}$$

$$\{a_{p\sigma}, a_{p'\sigma'}\} = \{a^\dagger_{p\sigma}, a^\dagger_{p'\sigma'}\} = 0 \qquad \forall p, p', \sigma, \sigma' \tag{2.10}$$

$$\{a_{p\sigma}, a^\dagger_{p'\sigma'}\} = 0 \qquad p \neq p' \tag{2.11}$$

$$\{a_{p\sigma}, a^\dagger_{p\sigma'}\} = 0 \qquad \sigma \neq \sigma' \tag{2.12}$$

$$[\hat{n}_{p\sigma}, a^\dagger_{p\sigma}] = a^\dagger_{p\sigma} \tag{2.13}$$

$$[\hat{n}_{p\sigma}, a_{p\sigma}] = -a_{p\sigma}. \tag{2.14}$$

The q-deformed number operator $\hat{n}_{p\sigma}$ in the above equations is equal to $(h_{p\sigma} + \frac{1}{2})$ only at the level of the physical representation. However, at the level of the algebra they are in general different.

Starting from equation (2.3) and using equations (2.9)–(2.14) it is straightforward but tedious to obtain

$$\mathcal{H}_0 = \frac{\epsilon}{4\sinh(\gamma/2)} \sinh\gamma \sum_p (h_{p\uparrow} - h_{p\downarrow}). \tag{2.15}$$

Until now we have been working with the deformed fermionic operators. The question now is to see if, as was done long ago in nuclear physics [20] for the non-deformed case, it is possible to construct a q-deformed quasi-spin algebra from the underlying q-deformed fermionic operators. Since our main interest is to study the collective excitations, in particular the phase transitions, we must construct the corresponding $su_q(2)$ quasi-spin Hamiltonian in order to recover the original simplicity of the SLM. We, therefore, introduce the operators

$$s_p^0 \equiv \tfrac{1}{2}(h_{p\uparrow} - h_{p\downarrow}) \tag{2.16}$$

$$s_p^+ \equiv a^\dagger_{p\uparrow} a_{p\downarrow} \tag{2.17}$$

$$s_p^- \equiv a^\dagger_{p\downarrow} a_{p\uparrow} \tag{2.18}$$

which satisfy the following commutation relations

$$[s_p^0, s_p^\pm] = \pm s_p^\pm \tag{2.19}$$

$$[s_p^+, s_p^-] = [2s_p^0]_q \tag{2.20}$$

where the bracket on the right-hand side of equation (2.20) is defined in the standard way

$$[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}} = \frac{e^{\gamma x} - e^{-\gamma x}}{e^\gamma - e^{-\gamma}}. \tag{2.21}$$

Furthermore, we define the q-deformed quasi-spin operators

$$S_0 \equiv \sum_p s_p^0 \tag{2.22}$$

$$S_\pm \equiv \sum_p e^{\gamma \sum_{p'<p} s_{p'}^0} s_p^\pm e^{-\gamma \sum_{p'>p} s_{p'}^0} = \sum_p e^{\gamma A_<(p)} s_p^\pm e^{-\gamma A_>(p)} \tag{2.23}$$

which satisfy the same commutation relations (2.19)–(2.20). It is important to emphasize the use of $h_{p\sigma}$ instead of $\hat{n}_{p\sigma}$, as usually performed in the literature in the bosonic case. As stated before, they are equivalent at the level of the representation.

The action of the S_0, $S_\pm$ operators on the deformed $|S, s\rangle$ basis [21] is given by

$$S_0|S, s\rangle = s|S, s\rangle \tag{2.24}$$

$$S_\pm|S, s\rangle = \sqrt{[S \mp s]_q [S \pm s + 1]_q}\,|S, s \pm 1\rangle \tag{2.25}$$

and, in particular, it is worth noting that for the Lipkin ground-state multiplet the basis states correspond to

$$|S, s\rangle = \left|\frac{N}{2}, s\right\rangle. \tag{2.26}$$

The q-deformed Lipkin Hamiltonian can then be rewritten in terms of S_0 and $S_\pm$ as

$$H = \frac{\epsilon}{4\sinh(\gamma/2)}\sinh(2\gamma S_0) + \frac{V}{2}(S_+^2 + S_-^2). \tag{2.27}$$

It is easy to verify that this expression goes back to the SLM Hamiltonian when $\gamma \to 0$ (or $q \to 1$), as do the commutation relations of the q-deformed quasi-spin operators.

We should point out here the difference between the Hamiltonian in equation (2.27) and the version of the Lipkin Hamiltonian used in previous papers [12, 14]. The difference lies in the mean-field term which now embodies q-deformation effects arising from a careful treatment of the q-deformation of the algebra already at the fermionic level.

3. The q-deformed Lipkin Hamiltonian in the $su_q(2)$ coherent states

Recently, in the works of Quesne and Jurčo [22] q-analogues of the $su(2)$ Perelomov coherent states [23] were defined for the $su_q(2)$ quantum algebra in terms of a q-exponential. Following [14] we define the q-analogue to the su (2) coherent state

$$|z\rangle \equiv e_q^{\bar{z}S_-}|S, s\rangle \tag{3.1}$$

where z is a complex number ($\bar{z}$ is its complex conjugate) which for later convenience will be parametrized as

$$z = \tan\frac{\theta}{2}e^{i\phi} \tag{3.2}$$

where $\theta \in [0, \pi]$, $\phi \in [0, 2\pi]$ and the q-exponential is

$$e_q^x = \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!} \tag{3.3}$$

with $[n]_q! = [n]_q[n-1]_q \ldots [1]_q$. We would like to note that our definition for the $su_q(2)$ coherent state is based on the maximum weight, whereas in [22] the minimal one was used.

Defining the q-binomial coefficient

$$\begin{bmatrix} n \\ m \end{bmatrix}_q \equiv \frac{[n]_q!}{[n-m]_q![m]_q!} \tag{3.4}$$

equation (3.1) can be rewritten as

$$|z\rangle = \sum_{s=-S}^{S} \begin{bmatrix} 2S \\ S-s \end{bmatrix}_q^{1/2} \bar{z}^{S-s}|S, s\rangle \tag{3.5}$$

whose normalization is given by

$$\langle z|z\rangle = \prod_{k=0}^{N-1}\left[1 + e^{\gamma(2k-N+1)}\right]. \tag{3.6}$$

We now use the coherent state (3.1) as a trial state for the q-deformed Lipkin Hamiltonian ground state. Thus, we get

$$\frac{\langle z|H|z\rangle}{\langle z|z\rangle} = \frac{\epsilon\langle z|\sinh(2\gamma S_0)|z\rangle}{4\sinh(\gamma/2)\langle z|z\rangle} + \frac{V\langle z|S_+^2 + S_-^2|z\rangle}{2\langle z|z\rangle} \tag{3.7}$$

where

$$\frac{\langle z|\sinh(2\gamma S_0)|z\rangle}{\sinh(\gamma/2)\langle z|z\rangle} = \frac{[N]_q}{[1/2]_q}\frac{\cos\theta}{\mathcal{D}(\gamma,\theta)} \tag{3.8}$$

$$\frac{\langle z|S_+^2 + S_-^2|z\rangle}{\langle z|z\rangle} = \frac{[N]_q[N-1]_q\sin^2\theta\cos 2\phi}{2\mathcal{D}(\gamma,\theta)} \tag{3.9}$$

and

$$\mathcal{D}(\gamma,\theta) = 1 + \sinh^2\left[\frac{\gamma}{2}(N-1)\right]\sin^2\theta. \tag{3.10}$$

We can normalize equation (3.7) as

$$E(\theta,\phi,\gamma,N) = \frac{\langle z|H|z\rangle}{\epsilon_q\langle z|z\rangle} = \frac{[N]_q}{2}\left\{\frac{\cos\theta}{\mathcal{D}(\gamma,\theta)} + \frac{\chi}{2}\frac{\sin^2\theta\cos 2\phi}{\mathcal{D}(\gamma,\theta)}\right\} \tag{3.11}$$

where

$$\epsilon_q = \frac{\epsilon}{2[1/2]_q}$$

is the q-deformed energy spacing and

$$\chi = \frac{V[N-1]_q}{\epsilon_q} \tag{3.12}$$

is an effective coupling strength.

4. Results and conclusions

Equation (3.11) is the expression for the variational energy from which we should extract the main information about the Lipkin model ground state, as described by the q-deformed coherent state. We should note that the energy depends on the deformation of the algebra and is proportional to $[N]_q$, whereas the terms enclosed by the curly brackets are functions of N and γ through the product $\gamma(N-1)$ and of the effective coupling strength χ.

In order to study the ground-state energy we must require the conditions

$$\frac{\partial E(\theta,\phi,\gamma,N)}{\partial\phi} = 0 \tag{4.1}$$

$$\frac{\partial E(\theta,\phi,\gamma,N)}{\partial\theta} = 0 \tag{4.2}$$

to be satisfied.

The first equation gives $\phi = \frac{\pi}{2}$ and $\frac{3\pi}{2}$ as global minima, the interesting physics lying on the interplay between θ and γN. The second equation exhibits two solutions with similar features as those obtained in the non-deformed Lipkin model, the first one being

$$\sin\theta = 0 \tag{4.3}$$

completely equivalent to the standard case, while the second one

$$-1 - \chi\cos\theta - 2C\cos\theta\frac{(\cos\theta - \frac{1}{2}\chi\sin^2\theta)}{1 + C\sin^2\theta} = 0 \tag{4.4}$$

where

$$C = \sinh^2\left[\frac{\gamma}{2}(N-1)\right] \tag{4.5}$$

now embodies the effects of the deformation of the algebra. Equation (4.4) is quadratic in $\cos\theta$ giving rise only to second-order phase transitions.

Equation (4.4) allows us to calculate the critical value of the strength parameter χ characterizing the phase transition whose analytical expression is then

$$\chi_c = 1 + 2\sinh^2\left[\frac{\gamma}{2}(N-1)\right]. \tag{4.6}$$

In the same fashion as discussed by Holzwarth [13], we would expect here the second-order phase transition, characterizing the spherical symmetry breaking in the quasi-spin space, to show up as the appearance of two symmetrical minima shifted from the origin and a maximum at the position of the old minimum. However, contrary to the standard case where $\chi_c = 1$, here the critical value of the coupling constant depends on the parameter $\gamma(N-1)$. This implies that now the phase transitions depend not only on the strength of the interaction but also on the deformation of the algebra and on the number of particles through the product $\gamma(N-1)$.

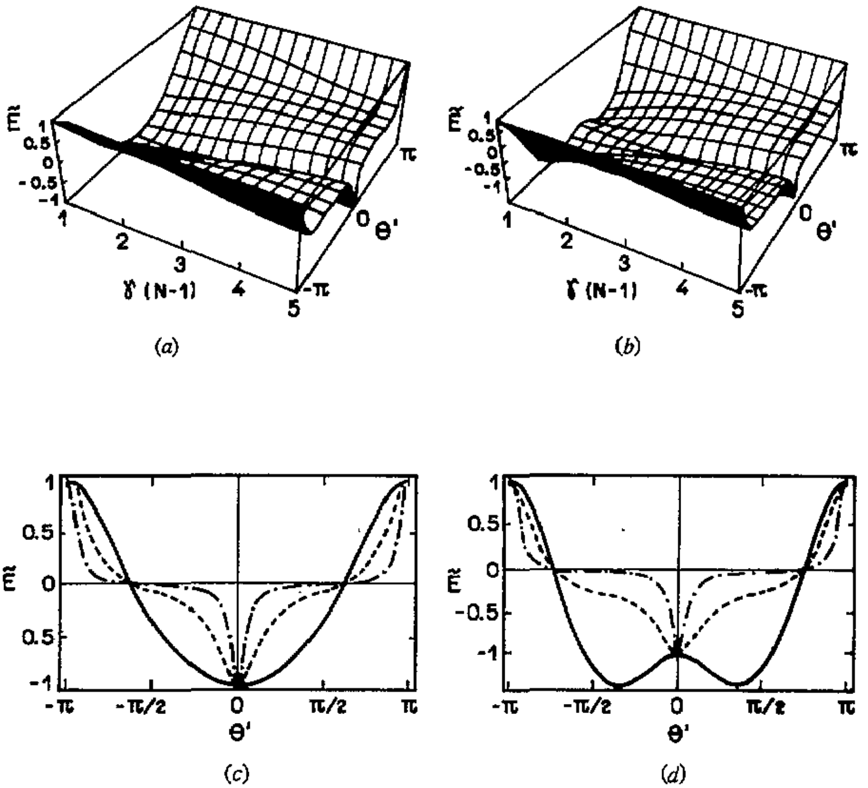

Figure 1. (a) and (b) show 3D views of the scaled energy surfaces ($\tilde{E} = 2E/[N]_q$) for $\chi = 1$ and 3 respectively, as a function of $\gamma(N-1)$ and of the order parameter $\theta' = \pi - \theta$. (c) and (d) show sections of the energy surfaces at $\gamma(N-1) = 1$ (full curve), 3 (dashed curve) and 5 (dot-dashed curve). The behaviour of $\tilde{E}$ for both global minima at $\varphi = \frac{\pi}{2}$ and $\frac{3\pi}{2}$ is shown together by extending the domain of θ' from $-\pi$ to π.

Figures 1(a) and (b) show scaled energy surfaces for different values of χ as a function of $\gamma(N-1)$ and the order parameter $\theta' = \pi - \theta$ [16], whereas figures 1(c) and (d) depict sections of the corresponding 3D-pictures for different values of $\gamma(N-1)$. There is a striking difference between the pictures on the left- and right-hand sides of figure 1, namely the number of minima. The reason for this behaviour in the first case is that χ_c, calculated by expression (4.6), is always greater than that for any value of $\gamma > 0$, as can be seen in figure 2. This in turn means that there will be no phase transition when one increases the deformation of the algebra for a fixed $\chi \leqslant 1$. Figures 1(b) and (d), however, present a gradual collapse of the two minima, characterizing the phase transition, in a new one at

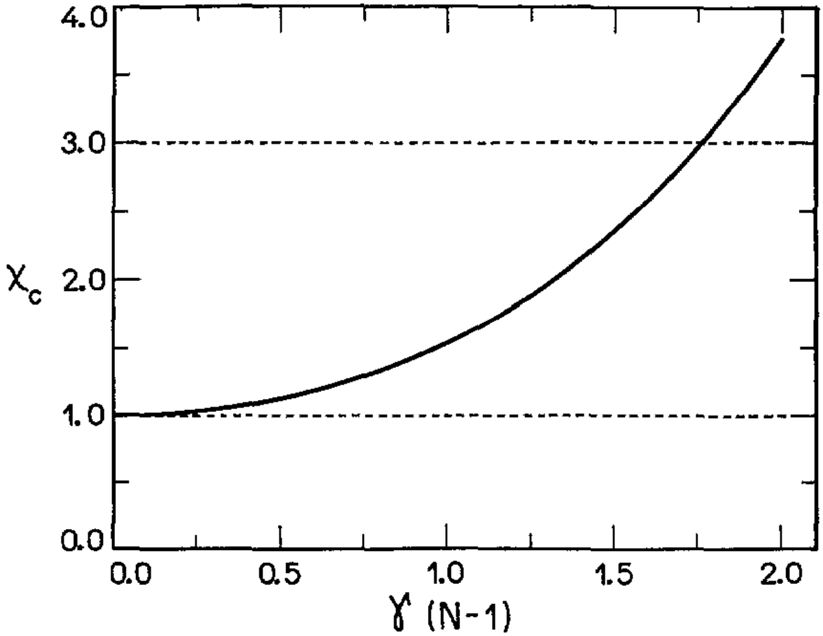

Figure 2. The critical value of χ as a function of $\gamma(N-1)$. The dashed curves indicate the region of existence of the phase transition.

$\theta' = 0$ as γ increases. For low values of $\gamma(N-1)$, $\chi = 3$ is greater than the value of χ_c, as can be seen from figure 2. In this range of $\gamma(N-1)$ we clearly identify the phase transition. However, for values of $\gamma(N-1)$ for which $\chi_c > 3$, no phase transition is allowed.

To summarize the main results of the present paper we would like initially to stress the importance of a careful treatment of the q-deformation, already at the fermionic level, in order to correctly take into account the effects in a many-body system. In the present case this gives rise to a q-dependent mean field as shown in equation (2.27). As a second aspect, we point out we have also obtained a critical value of χ, equation (4.6), which is a function of $\gamma(N-1)$. This means that a universal character can no longer be assigned to χ as a system-independent indicator of the phase transition in a q-deformed system.

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